COMPLEX ANALYSIS
TOPIC XII: CAUCHY'S INTEGRAL FORMULA

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1. Review

Definition 1. Let D be an open subset of $\mathbb{C}$ and let $f : D \to \mathbb{C}$. We say that f is *analytic* at $z_0 \in D$ if f is differentiable at every point in a neighborhood of z_0. We say that f is analytic on D if f is differentiable at every point in D.

Definition 2. Let D be an open subset of $\mathbb{C}$ and let $f : D \to \mathbb{C}$ be analytic. Let $\gamma : [a, b] \to D$ be a piecewise smooth path. The *path integral of f along γ* is

$$\int_\gamma f(z)\, dz = \int_a^b f(\gamma(t))\gamma'(t)\, dt.$$

Theorem 1. *Let D be an open connected set and let $f : D \to \mathbb{C}$. Suppose that f admits a primitive F in D. Let $z_1, z_2 \in D$. Then for every piecewise smooth path $\gamma : [a, b] \to D$ with $\gamma(a) = z_1$ and $\gamma(b) = z_2$,*

$$\int_\gamma f(z)\, dz = F(z_2) - F(z_1).$$

Theorem 2. *Let D be an open subset of $\mathbb{C}$. Let $f : D \to \mathbb{C}$ be analytic on D. Let α and β be homotopic paths in D. Then*

$$\int_\alpha f(z)\, dz = \int_\beta f(z)\, dz.$$

Corollary 1. *Let D be an open, connected, simply connected subset of $\mathbb{C}$. Let f be analytic on D. Then for every simple closed curve C in D,*

$$\int_C f(z)\, dz = 0.$$

Date: January 29, 2018.

2. Technical Lemmas

To continue our development, we need a bound on the size of an integral.

Lemma 1. *Let $\alpha : [a, b] \to \mathbb{C}$ be a piecewise smooth path in $\mathbb{C}$. The arclength of α is*

$$L = \int_a^b |\alpha'(t)| \, dt.$$

Proof. We partition the domain of α into n pieces of equal size, by setting

$$\Delta t = \frac{b-a}{n} \qquad \text{and} \qquad t_i = a + i\Delta t.$$

An estimate for the length of the arc is

$$L \approx \sum_{i=1}^{n} |\alpha(t_i) - \alpha(t_{i-1}| = \sum_{i=1}^{n} \Big| \frac{\alpha(t_i) - \alpha(t_{i-1})}{\Delta t} \Big| \Delta t.$$

Taking the limit as $n \to \infty$ we obtain

$$L = \lim_{\Delta t \to 0} \sum_{i=1}^{n} \Big| \frac{\alpha(t_i) - \alpha(t_{i-1})}{\Delta t} \Big| \Delta t = \int_a^b |\alpha'(t)| \, dt.$$

□

Lemma 2. *Let $\alpha : [a, b] \to \mathbb{C}$ be a piecewise smooth path in $\mathbb{C}$. Then*

$$\Big| \int_a^b \alpha(t)\,dt \Big| \le \int_a^b |\alpha(t)|\,dt.$$

Proof. This is certainly true if the integral on the left hand side is zero, so let us assume that it is not; thus, set $re^{i\theta} = \int_a^b \alpha(t)\,dt$. Then $|\int_a^b \alpha(t)\,dt| = r$, and

$$\begin{aligned}
r &= \int_a^b e^{-i\theta}\alpha(t)\,dt && \text{since } e^{i\theta} \text{ is constant} \\
&= \mathrm{Re}\Big(\int_a^b e^{-i\theta}\alpha(t)\,dt \Big) && \text{since the integral is real} \\
&= \int_a^b \mathrm{Re}(e^{-i\theta}\alpha(t))\,dt && \text{by the definition of the integral} \\
&\le \int_a^b |e^{-i\theta}\alpha(t)|\,dt && \text{since } \mathrm{Re}(z) \le |z| \text{ for } z \in \mathbb{C} \\
&= \int_a^b |e^{-i\theta}||\alpha(t)|\,dt && \text{by a property of modulus} \\
&= \int_a^b |\alpha(t)|\,dt && \text{since } |e^{-i\theta}| = 1
\end{aligned}$$

□

Lemma 3. *Let D be an open subset of $\mathbb{C}$ and let $f : D \to \mathbb{C}$ be continuous. Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise smooth path in D. Suppose that L is the arclength of γ, and that suppose $f(z) \le M$ for all $z \in D$. Then*

$$\Big| \int_\gamma f(z)\,dz \Big| \le ML.$$

Proof. Behold:

$$\begin{aligned}
\Big| \int_\gamma f(z)\,dz \Big| &\le \int_\gamma |f(z)|\,dz && \text{by Lemma 2} \\
&= \int_\gamma |f(\gamma(t))\gamma'(t)|\,dt && \text{by the definition of path integration} \\
&= \int_\gamma |f(\gamma(t))||\gamma'(t)|\,dt && \text{by a property of modulus} \\
&\le \int_\gamma M|\gamma'(t)|\,dt && \text{by a property of Riemann integration} \\
&= M \int_\gamma |\gamma'(t)|\,dt && \text{by a property of Riemann integration} \\
&= ML && \text{by Lemma 1}
\end{aligned}$$

□

3. Antiderivatives

Theorem 3. *Let D be an open connected subset of $\mathbb{C}$, and let $f : D \to \mathbb{C}$ be continuous. The following are equivalent.*

(a) *The function f admits a primitive in D.*
(b) *The integral of f between two points in D is path independent.*
(c) *The integral of f along a closed curve in D is zero.*

Proof. We have previously seen why **(b)** and **(c)** are equivalent, and that **(a)** implies **(b)**. It remains to see why **(b)** implies **(a)**.

Suppose that integration of f in D is path independent. That is, for any $z_1, z_2 \in D$ and any paths α and β from z_1 to z_2, we have $\int_\alpha f(z)\,dz = \int_\beta f(z)\,dz$. Thus we write such an integral as $\int_{z_1}^{z_2} f(z)\,dz$.

Select $z_0 \in D$. Note that since D is connected, there is a path from z_0 to every $z \in D$. Define a function

$$F : D \to \mathbb{C} \quad \text{by} \quad F(z) = \int_{z_0}^{z} f(s)\,ds;$$

the value of F is independent of the path we trace from z_0 to z.

Let $z \in D$. Since D is open, there is an $\delta > 0$ such that $B_\delta(z) \subset D$. Let $h \in \mathbb{C}$ with $|h| < \delta$. Then the line segment from z to $z + h$ is contained in D. Let $\alpha : [0,1] \to D$ given by $\alpha(t) = z + ht$ parameterize this line segment. Then $\alpha'(t) = h$, so

$$\int_\alpha ds = \int_0^1 h\,dt = h.$$

Let ϵ be any positive real number. Since f is continuous, we can select δ above such that $|s - z| < \delta$ implies $|f(s) - f(z)| < \epsilon$. Then, by Lemma 3,

Now, by path independence,

$$F(z+h) - F(z) = \int_{z_0}^{z+h} f(s)\,ds - \int_{z_0}^{z} f(s)\,ds = \int_{z}^{z+h} f(s)\,ds.$$

Since $\int_z^{z+h} ds = h$ and $f(z)$ is constant with respect to s, we have

$$f(z) = \frac{f(z)}{h} \int_z^{z+h} ds = \frac{1}{h} \int_z^{z+h} f(z)\,ds,$$

so that

$$\frac{F(z+h) - F(z)}{h} - f(z) = \frac{1}{h} \int_z^{z+h} (f(s) - f(z))\,ds.$$

Let ϵ be any positive real number. Since f is continuous, we can select δ above such that $|s - z| < \delta$ implies $|f(s) - f(z)| - < \epsilon$. Then, by Lemma 3,

$$\frac{F(z+h) - F(z)}{h} - f(z) < \frac{1}{h}(|h|\epsilon),$$

so that

$$\Big| \frac{F(z+h) - F(z)}{h} - f(z) \Big| < \epsilon.$$

Since ϵ is arbitrary,

$$\lim_{h \to 0} \Big| \frac{F(z+h) - F(z)}{h} - f(z) \Big| = 0.$$

We conclude that $F'(z) = f(z)$. □

4. Winding Numbers

The Jordan Curve Theorem indicates that a simple closed curve in a plane partitions its complement into two components, an inside and an outside. Considering that the curve can be very complex, this turns out to be more difficult to prove than expected. The notions of the homotopy and winding numbers give alternate formulations of inside and outside.

Proposition 1. *Let $z_0 \in \mathbb{C}$ and let C be a circle of radius 1 centered at z_0, with positive orientation. Then*

$$\int_C \frac{dz}{z - z_0} = 2\pi i.$$

Proof. Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be given by $\gamma(t) = z_0 + e^{it}$. Then γ is a parameterization of C, and $\gamma'(t) = ie^{it}$, so

$$\int_C \frac{dz}{z - z_0} = \int_0^{2\pi} \frac{1}{\gamma(t) - z_0}\gamma'(t)\, dt = \int_0^{2\pi} \frac{ie^{it}}{(z_0 + e^{it}) - z_0}\, dt = i\int_0^{2\pi} dt = 2\pi i.$$

□

The function $f(z) = \dfrac{1}{z - z_0}$ may be used to give information about curves. It is clear that if we go around the circle C centered at z_0 twice, the integral $\int_C f(z)\, dz$ will be $4\pi i$, and if we go around once in the opposite direction, the integral will be $-2\pi i$; in general, if we go around the circle k times, the integral will be $2k\pi i$. Moreover, any loop γ which is homotopic to a circular loop which wraps k times around z_0 k will have $\int_\gamma f(z)\, dz = 2k\pi i$.

Definition 3. Let $\gamma : I \to \mathbb{C}$ be a loop in $\mathbb{C}$, and let $z_0 \in \mathbb{C}$. The *winding number* (or *index*) of γ about z_0 is

$$n(\gamma, z_0) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z - z_0}\, dz.$$

The domain of analyticity of $f(z) = \dfrac{1}{z - z_0}$ is $D = \mathbb{C} \smallsetminus \{z_0\}$. It is intuitively clear that any simple closed curve in $\mathbb{C}$ which does not pass through z_0 is homotopic to a circle in D. If z_0 is inside such a circle, the curve is further homotopic to a circle of radius one centered at z_0, in which case the integral of $f(z)$ around the circle is $2\pi i$. On the other hand, if z_0 is not inside the circle, the curve is homotopic to a constant in D, so the integral is zero. Given a simple closed curve $C \subset \mathbb{C}$ and a point $z_0 \in \mathbb{C} \smallsetminus C$, we have these methods to detect whether the point is inside the curve.

- z_0 is an element of the bounded component of $\mathbb{C} \smallsetminus C$;
- a (nontangential) ray from z_0 to ∞ intersects the curve an odd number of times;
- C is homotopic to a constant in $\mathbb{C} \smallsetminus \{z_0\}$;
- $n(\gamma, z_0) = \pm 1$.

If C is a simple closed curve, then the winding number tells whether z_0 is inside $n(\gamma, z_0) = \pm 1$ or outside $n(\gamma, z_0) = 0$ the curve. However, if C is the image of any loop, $n(\gamma, z_0)$ tells how many times C wraps around z_0. The situation regarding "inside" or "outside" is less clear if the curve is not simple. Indeed, we could define z_0 to be outside of a closed curve if the curve is homotopic to a constant in $\mathbb{C} \setminus \{z_0\}$. However, it turns out that there exist loops γ such that $n(\gamma, z_0) = 0$, and yet γ is not homotopic to a constant in $\mathbb{C} \setminus \{z_0, z_1\}$, where z_1 is some other point.

5. Rational Functions

To review, we note that one can compute the next proposition directly from the definition.

Result 1. (Logarithms) Let C be a positively oriented circle centered at $z_0 \in \mathbb{C}$. Then

$$\int_C \frac{1}{z - z_0}\, dz = 2\pi i.$$

The following extension of this fact follows from the fact that $f(z) = \dfrac{1}{(z - z_0)^n}$ has an antiderivative in its domain, if $n > 1$.

Result 2. (Power functions) Let C be a positively oriented circle centered at $z_0 \in \mathbb{C}$, and let $k \in \mathbb{Z}$. Then

$$\int_C (z - z_0)^k\, dz = \begin{cases} 2\pi i & \text{if } k = -1\ ; \\ 0 & \text{if } k \neq -1\ . \end{cases}$$

In particular,

Result 3. (Polynomials) Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial, and let C be any closed curve in $\mathbb{C}$. Then

$$\int_C f(z)\, dz = 0.$$

Combine these with homotopy, winding numbers, and the principle of partial fraction decomposition to obtain the following.

Result 4. (Rational Functions) Let $f(z)$ be a rational function with poles $p_1, p_2, \ldots, p_n$ of order $o_1, o_2, \ldots, o_n$, respectively. Using partial fractions, we may write

$$f(z) = g(z) + \sum_{j=1}^{n} \sum_{k=1}^{o_j} \frac{A_{jk}}{(z - z_j)^k},$$

where $g(z)$ is a polynomial and $A_{jk} \in \mathbb{C}$. Let C be an closed curve in $\mathbb{C}$. Then

$$\int_C f(z)\, dz = 2\pi i \sum_{j=1}^{n} A_{j1} n(C, z_j),$$

where

$$n(C, z_j) = \frac{1}{2\pi i} \int_C \frac{1}{z - z_j}\, dz$$

is the winding number of C about z_j.

6. Cauchy's Integral Formula

The technique used to show that having an antiderivative is equivalent to path independence may be modified to provide the following striking integration formula. We begin with a lemma, which really contains the main argument.

Theorem 4. (Cauchy's Integral Formula) *Let f be analytic on and inside a simple closed curve C with positive orientation. Let z_0 be inside of C. Then*

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz.$$

Proof. Because of path independence within C, we may assume that C is a circle of radius ρ centered at z_0. Thus the length of C is $L = 2\pi\rho$.

Let ϵ be any positive real number and let $M = \frac{\epsilon}{L}$. Since f is differentiable at z_0, we can select δ such that $|z - z_0| < \delta$ implies $\left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < M$. Then, by Lemma 3,

$$\left| \int_C \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\, dz \right| \leq LM = \epsilon.$$

Since ϵ is arbitrary, we see that

$$\int_C \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\, dz = 0,$$

so

$$\int_C \frac{f(z) - f(z_0)}{z - z_0}\, dz = \int_C f'(z_0)\, dz = 0,$$

since f' has a primitive inside C. This shows that

$$\int_C \frac{f(z)}{z - z_0}\, dz = \int_C \frac{f(z_0)}{z - z_0}\, dz = f(z_0) \int_C \frac{dz}{z - z_0} = \frac{f(z_0)}{2\pi i}.$$

□

Theorem 5. (Cauchy's Integral Extension) *Let f be analytic on and inside a simple closed curve C with positive orientation. Let z_0 be inside of C. Let n be a positive integer. Then the n^{th} derivative of f at z_0 exists, and*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz.$$

Reason. Under the circumstances of this theorem, for reasons we do not investigate, it is possible to "differentiate under the integral sign". That is, we apply the differentiation operator $\frac{d}{dz}$ to each side of Cauchy's Integral Formula and obtain

$$\begin{aligned} f'(z_0) &= \frac{d}{dz} \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz \\ &= \frac{1}{2\pi i} \int_C \frac{d}{dz} \left[\frac{f(z)}{z - z_0} \right] dz \\ &= \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^2}\, dz \end{aligned}$$

Consider this the base case; the form for $n > 1$ follows similarly using induction. □

7. Morera's Theorem

Morera's Theorem can be considered to be a converse of the Cauchy-Goursat theorem.

Theorem 6. (Morera's Theorem) *Let $D \subset \mathbb{C}$ be an open set and let $f : D \to \mathbb{C}$ be continuous. Suppose that*

$$\int_C f(z)\, dz = 0$$

for every simple closed contour C in D. Then f is analytic on D.

Proof. This follows from Theorem 3 and Theorem 5 as follows.

Since the integral around every closed curve is zero, we know that integration of f in D is independent of path. Thus we can construct an antiderivative F for f in D. Now F is differentiable at every point in D, since $F'(z) = f(z)$ for all $z \in D$. Thus F is analytic in D, and now Cauchy's Integral Extension implies that $F' = f$ is also differentiable. □

8. Liouville's Theorem

Lemma 4. (Cauchy's Estimate) *Let f be analytic within and on a the circle C of radius R about $z_0 \in \mathbb{C}$. Let M be an upper bound for the modulus of f on C. Then*

$$|f^{(n)}(z_0)| \leq \frac{n!M}{2\pi R^n}.$$

Proof. By Cauchy's Integral Extension,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz.$$

Taking the modulus of both sides gives

$$\begin{aligned}
|f^{(n)}(z_0)| &= \left| \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz \right| && \text{taking the modulus of both sides} \\
&\leq \frac{n!}{2\pi} \int_C \frac{|f(z)|}{|(z - z_0)^{n+1}|}\, dz && \text{by Lemma 2} \\
&= \frac{n!}{2\pi R^{n+1}} \int_C |f(z)|\, dz && \text{since } |z - z_0| = R \text{ on } C \\
&\leq \frac{n!}{2\pi R^{n+1}} (2\pi R) M && \text{by Lemma 1} \\
&= \frac{n!M}{2\pi R^n}
\end{aligned}$$

□

Theorem 7. (Liouville's Theorem) *A bounded entire function is constant.*

Proof. Let $f : \mathbb{C} \to \mathbb{C}$ be a bounded entire function, say $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Let $z_0 \in \mathbb{C}$, and let R be the radius of a circle about z_0. With $n = 1$, Lemma 4 says that

$$f'(z_0) \leq \frac{M}{2\pi R^n}$$

for all $R > 0$. This implies that $f'(z_0) = 0$. Since z_0 was arbitrary, $f'(z) = 0$ for all $z \in \mathbb{C}$. Therefore, f is constant. □

9. Fundamental Theorem of Algebra

Theorem 8. (Fundamental Theorem of Algebra) *Let $f : \mathbb{C} \to \mathbb{C}$ be a nonconstant polynomial function. Then there exists $z \in \mathbb{C}$ such that $f(z) = 0$.*

Proof. We assume that f is a nonconstant polynomial with no zeros, and arrive at a contraction For simplicity and without loss of generality, assume f is monic.

Since f is nonconstant, f becomes unbounded in every direction as z goes to infinity. To see this, write $f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$. Then

$$\lim_{z\to\infty} f(z) = \lim_{z\to\infty} z^n(1 + \frac{a_{n-1}}{z} + \cdots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n}) = \lim_{z\to\infty} z^n = \infty.$$

Consider the function $g(z) = \dfrac{1}{f(z)}$. Since f has no zeros, g is defined and analytic in the entire complex plane. Since $\lim_{z\to\infty} f(z) = \infty$, we see that $\lim_{z\to\infty} g(z) = 0$. Thus there exists $R > 0$ such that, for $|z| > R$, we have $|f(z)| < 1$. Since modulus is a continuous function, f attains a maximum modulus on the closed disk given by $|z| \leq R$. Thus f is bounded, and so by Liouville's Theorem, f is constant, contradicting our hypothesis. □

10. Maximum Modulus Theorem

Lemma 5. *Let $f : [a, b] \to \mathbb{R}$ be continuous. Let*

$$M = \frac{1}{b-a}\int_a^b f(t)\,dt.$$

Suppose that $f(t) \leq M$ for all $t \in [a, b]$. Then $f(t) = M$ for all $t \in [a, b]$.

Proof. Suppose not. Then $f(t_0) \neq M$ for some $t_0 \in [a, b]$. Thus $f(t_0) < M$. Let $\epsilon = \dfrac{M - f(t_0)}{2}$. Since f is continuous, there exists $\delta > 0$ such that $|t - t_0| < \delta$ implies $|f(t) - f(t_0)| < \epsilon$. Let $c = t_0 - \delta$ and $d = t_0 + \delta$, so that for $t \in (c, d)$, we have $t < M - \epsilon$. Then $\int_c^d f(t)\,dt \leq (d - c)(M - \epsilon)$.

Since $|f(t)| \leq M$ for all $t \in [a, b]$, we have $\int_a^c f(t)\,dt \leq (c-a)M$, and $\int_d^b f(t)\,dt \leq (b - d)M$. So

$$\begin{aligned}(b-a)M &= \int_a^b f(t)\,dt \\ &= \int_a^c f(t)\,dt + \int_c^d f(t)\,dt + \int_d^b f(t)\,dt \\ &\leq (c-a)M + (d-c)(M-\epsilon) + (b-d)M \\ &< (b-a)M\end{aligned}$$

This contradiction proves the lemma. □

Lemma 6. *Let $f : [a, b] \to \mathbb{C}$ be continuous. Let*

$$\mu = \frac{1}{b-a} \int_a^b f(t)\, dt.$$

Suppose that $|f(t)| \le |\mu|$ for all $t \in [a, b]$. Then $f(t) = \mu$ for all $t \in [a, b]$.

Proof. Dividing my μ gives

$$1 = \frac{1}{b-a} \int_a^b \frac{f(t)}{\mu}\, dt.$$

Let $g(t) = \mathrm{Re}(f(t)/\mu)$ and $h(t) = \mathrm{Im}(f(t)/\mu)$, so that $\dfrac{f(t)}{\mu} = g(t) + ih(t)$. Then

$$1 = \frac{1}{b-a} \int_a^b g(t)\, dt + \frac{i}{b-a} \int_a^b h(t)\, dt.$$

Since the left hand side is real, the imaginary part of the right hand side is zero, so

$$1 = \frac{1}{b-a} \int_a^b g(t)\, dt.$$

Taking the modulus of both sides gives

$$1 = \frac{1}{b-a} \left| \int_a^b g(t)\, dt \right| \le \frac{1}{b-a} \int_a^b |g(t)|\, dt.$$

But since $|f(t)| \le \mu$ for all $t \in [a, b]$,

$$|g(t)| = |f(t)/\mu| \le 1 \quad \text{for all } t \in [a, b].$$

Thus Lemma 5 implies that $g(t) = 1$ for all $t \in [a, b]$. Since $|f(t)/\mu| = |g(t)+ih(t)| \le 1$, we must have $h(t) = 0$ for all $t \in [a, b]$. Thus $\dfrac{f(t)}{\mu} = g(t) = 1$, so $f(t) = \mu$, for all $t \in [a, b]$. □

Lemma 7. *Let $z_0 \in \mathbb{C}$ and let D be an open disk centered at z_0. Let f be a function which is analytic on D, such that $|f(z)| < |f(z_0)|$ for all $z \in D$. Then $f(z) = f(z_0)$ for all $z \in D$.*

Proof. Let r be a positive real number which is less than the radius of D, and let $\gamma : [0, 2\pi] \to D$ be given by $\gamma(t) = z_0 + re^{it}$. We apply Cauchy's Integral Formula to see that

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz \\ &= \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\, dz \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(\gamma(t))}{\gamma(t) - z_0} \gamma'(t)\, dt \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{it})}{z_0 + re^{it}} (rie^{it})\, dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\, dt. \end{aligned}$$

By hypothesis, $|f(z_0 + re^{it})| \leq |f(z_0)|$ for all $t \in [0, 2\pi]$, so by Lemma 6, we have $f(z) = f(z_0)$ for all z in the disk about z_0 of radius r. Since may be r is arbitrarily close to the radius of D, this implies that $f(z) = f(z_0)$ for all $z \in D$. □

Theorem 9. (Maximum Modulus Theorem) *Let D be an open connected subset of $\mathbb{C}$ and let $f : D \to \mathbb{C}$ be analytic. If there exists $z_0 \in \mathbb{C}$ such that $f(z) \leq f(z_0)$ for every $z \in D$, then f is constant.*

Proof. For $w \in \mathbb{C}$ and $r \in \mathbb{R}$, let

$$B_r(w) = \{z \in \mathbb{C} \mid |z - w| < \delta$$

denote the open disk of radius r about w.

Let $w \in D$. We show that $f(w) = f(z_0)$.

Since D is connected, there exists a contour C from z_0 to w. Let $r > 0$ be so small that a disk of radius r about each point on C is contained in D. We can pick a finite number of points $z_0, z_1, \ldots, z_n = w$ on C such that $|z_i - z_{i-1}| < r$ for $i = 1, \ldots, n$. By Lemma 7, $f(z) = f(z_0)$ for all $z \in B_r(z_0)$, and since $z_1 \in B_r(z_0)$, $f(z_1) = f(z_0)$. Since $B_r(z_1) \subset D$, we still have $f(z) \leq f(z_1)$ for all $z \in B_r(z_1)$, so f is constant on $B_r(z_1)$. Similarly, $z_2 \in B_r(z_1)$, so $f(z_2) = f(z_1)$, so f has a maximum modulus at z_2. Continuing in this way, by induction, we see that

$$f(z_0) = f(z_1) = \cdots = f(z_n) = f(w).$$

Thus f is constant on all of D. □

Corollary 2. *Let D be an open connected subset of $\mathbb{C}$ and let $f : D \to \mathbb{C}$ be analytic and nonvanishing. Suppose that $f(z) \neq 0$ for all $z \in D$. If there exists $z_0 \in \mathbb{C}$ such that $f(z) \geq f(z_0)$ for every $z \in D$, then f is constant.*

Proof. Since f is nonvanishing, $f(z) \neq 0$ for all $z \in D$, so $\dfrac{1}{f(z)}$ is analytic in D. Then $\dfrac{1}{f(z_0)}$ has a maximal modulus in D, which implies that $\dfrac{1}{f(z)}$ is constant, and so is $f(z)$. □

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